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BOUNDS FOR THE FREE VIBRATION FREQUENCIES OF HOMOGENEOUS ANISOTROPIC BODIES WITH CONSTRAINED BOUNDARY†

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A modification of the classical comparison theorem for the free vibration frequencies of homogeneous linearly elastic bodies of an arbitrary anisotropy, which occupy a region of arbitrary shape with clamped boundary, is proved by means of Van Hove's theorem. Some other similar modifications of the comparison theorem for homogeneous linearly elastic bodies of special types of anisotropy (characterized by the presence of specular symmetry), having the shape of a rectangular parallelepiped with faces parallel to the planes of symmetry, and with sliding boundary conditions either along the faces or along their normals, are proved using modifications of Van Hove's theorem. On the basis of the set of proved modifications of the comparison theorem, a method for obtaining refined bilateral bounds for all frequencies of the free vibration spectrum pertinent to the specified problems (for which the exact values of frequencies are, as a rule, unknown) is proposed. The bounds turn out to depend in a simple manner on the least and the greatest velocities of propagation of elastic waves in a solid and on the characteristic geometrical dimensions of the body. Examples are considered. A version of the comparison theorem modifications and a method of obtaining the bounds for frequencies, suitable for the linearized problem of small free vibrations of homogeneous uniformly strained non-linearly elastic bodies, and also for free vibrations of moderately inhomogeneous linearly elastic ones, is proposed.

The theory of free vibrations of linearly elastic bodies reduces, in mathematical respects, to the eigenvalue theory for a certain type of linear self-adjoint operators in Hilbert space (see for example [1–3]); the general properties of both the spectrum and the system of eigenvectors of such operators have been studied extensively. Nevertheless, the number of problems on free vibrations where exact solutions and correspondingly exact values of the characteristic frequencies are known is extremely small; all of them concern the case of a Hookean material and bodies of simple shape: a sphere, a rectangular parallelepiped, etc. In other problems, where the exact values of the frequencies are unknown, they can be replaced by rigorous bilateral bounds, the lower bound for the lowest frequency being of particular interest.

In the general theory [2, 3] the classical comparison theorem on the characteristic frequencies of different bodies is proved, which enables one, in principle, to obtain bilateral bounds for frequencies when Hookean bodies, which occupy regions of simple shape (since their frequencies are known), are used for comparison.

In the present paper, for a particular class of linear free vibration problems we prove some modifications of the comparison theorem with a less restrictive inequality imposed on the elasticity tensors of the bodies whose frequencies are compared with one another. The particular class of problems is specified by the conditions of Van Hove's theorem [4] and its modifications [5–7], the use of which leads to a less restrictive inequality.

Modifications of the comparison theorem enable one to obtain refined (as compared to the classical theorem) bounds for the frequencies, which are moreover simple and physically clear. In certain cases the refinement is significant. In the paper we consider the example of a body with an extremely simple orthotropic elastic relation, occupying a region that is flattened considerably in a certain direction. The lower bound for the lowest characteristic frequency of such a body, resulting from the classical theorem, is found to be many times less than the exact value, whereas the analogous bound resulting from the modification of the theorem, is almost identical with it. This example, together with the others, demonstrates the efficiency of the method for constructing bounds resulting from the proposed theory.

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1. A BRIEF SUMMARY OF WELL-KNOWN RESULTS ON THE FREE VIBRATIONS OF LINEARLY ELASTIC BODIES WHICH ARE USED BELOW

We use Gibbs' system of tensor notation, supplemented with the tensor-product sign and with a right upper multi-index to denote isomers of a tensor. In the formulae directly related to free vibrations, the notation employed is close to that used in [3], which serves as a basis for the present statement of the main available results.

In stating the formulae of the theory of free vibrations of linearly elastic bodies we will confine ourselves to the special cases considered in the paper: the elastic body is homogeneous, there are no body forces, the boundary conditions are either those of zero displacements on the whole of the boundary (a "clamped" boundary), or it is assumed that there are plane portions of the boundary and the conditions of free sliding, either along the planes themselves, or along their normals, are specified on them, while the remainder of the boundary is clamped; there are no free portions of the boundary.

Let us suppose that the body in the equilibrium configuration, taken as a reference configuration [8], occupies the region B of three-dimensional Euclidean space. The material points of the body are identified by means of their position vectors $\mathbf{x}$ in this configuration.

We denote by $\mathbf{u}(\mathbf{x}, t)$ the displacement field at time t, by $\nabla \otimes \mathbf{u}(\mathbf{x}, t)$ the displacement gradient field ($d\mathbf{u} = d\mathbf{x} \cdot \nabla \otimes \mathbf{u}$) and, by $\mathbf{T}(\mathbf{x}, t)$ the stress tensor field (generally speaking, it is the Piola stress tensor [8], which, within the framework of the linear theory, is identical with the Cauchy tensor). The linear elastic governing relation is specified in the form

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{C}:\nabla\otimes\mathbf{u}(\mathbf{x}, t) \tag{1.1}$$

where $\mathbf{C}$ is the elasticity tensor (of the fourth rank). Since the body is assumed to be homogeneous, the tensor $\mathbf{C}$ has the same value at all its points.

Let us introduce a notation for the fourth-order tensor isomers. Let $(i_1i_2i_3i_4)$ be a permutation of the set (1234). Then for reducible tensors (tetrads)

$$\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4^{(i_1i_2i_3i_4)} = \mathbf{a}_{i_1} \otimes \mathbf{a}_{i_2} \otimes \mathbf{a}_{i_3} \otimes \mathbf{a}_{i_4} \tag{1.2}$$

and for arbitrary tensors $\mathbf{C}^{(i_1i_2i_3i_4)}$ denotes that each tetrad in a representation of the tensor $\mathbf{C}$ is replaced by its corresponding isomer according to (1.2); in other words, if $(j_1j_2j_3j_4)$ is the inverse permutation for $(i_1i_2i_3i_4)$, then for any four vectors $\mathbf{u}_1, \ldots, \mathbf{u}_4$

$$\begin{aligned}\mathbf{u}_1 \otimes \mathbf{u}_2 : \mathbf{C}^{(i_1i_2i_3i_4)} : \mathbf{u}_3 \otimes \mathbf{u}_4 &= \mathbf{C}^{(i_1i_2i_3i_4)} \vdots\, \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3 \otimes \mathbf{u}_4 = \\ &= \mathbf{C} \vdots\, \mathbf{u}_{j_1} \otimes \mathbf{u}_{j_2} \otimes \mathbf{u}_{j_3} \otimes \mathbf{u}_{j_4} = \mathbf{u}_{j_1} \otimes \mathbf{u}_{j_2} : \mathbf{C} : \mathbf{u}_{j_3} \otimes \mathbf{u}_{j_4}\end{aligned} \tag{1.3}$$

Due to the presence of an elastic potential, both in the classical linear theory and in linearized non-linear one, the elasticity tensor is invariant under "pairwise permutation"

$$\mathbf{C}^{(3412)} = \mathbf{C} \tag{1.4}$$

In the classical linear theory there is one more symmetry

$$\mathbf{C}^{(1243)} = \mathbf{C}^{(2134)} = \mathbf{C}$$

that does not occur in the linearized non-linear theory. In what follows, this symmetry is never employed, and hence, the range of applicability of the results is not restricted to the cases where it occurs.

Let us write the equation of motion of an elastic body for the interior of occupied region B, denoting, as is usually done, the material derivative with respect to time by a dot above a symbol

$$\rho\ddot{\mathbf{u}}(\mathbf{x}, t) = \nabla \cdot \mathbf{T}(\mathbf{x}, t) = \nabla \cdot \mathbf{C} : \nabla \otimes \mathbf{u}(\mathbf{x}, t) \tag{1.5}$$

where $\rho = \text{const}$ is the density of the body in the reference configuration and $\nabla \cdot \mathbf{T}$ is the stress tensor field divergence with respect to this configuration ($\nabla \cdot \mathbf{T} = \mathbf{I}: \nabla \otimes \mathbf{T}$, where $\mathbf{I}$ is the unit second-rank tensor).

Let Σ^0 be the clamped portion of the boundary ∂B, Σ' the set of plane portions with sliding along the plane Σ'' the set of plane portions with sliding along the normal $\mathbf{n}$ the outward normal to the boundary.

The boundary conditions then take the form

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Sigma^0 \\
&\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot \mathbf{T}(\mathbf{x}, t) \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = 0, \quad \mathbf{x} \in \Sigma^t \\
&\mathbf{u}(\mathbf{x}, t) \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = 0, \quad \mathbf{n} \cdot \mathbf{T}(\mathbf{x}, t) \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \Sigma^n
\end{aligned} \tag{1.6}$$

We will use the term free vibrations for the displacement fields that vary sinusoidally with time and obey the equation of motion and the boundary conditions

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x})\sin(\omega t + \gamma) \\
&\nabla \cdot \mathbf{C}{:}\nabla \otimes \mathbf{u}(\mathbf{x}) + \rho\omega^2\mathbf{u}(\mathbf{x}) = 0 \\
&\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Sigma^0 \\
&\mathbf{u}(\mathbf{x}) \cdot \mathbf{n} = 0, \quad (\mathbf{n} \cdot \mathbf{C}{:}\nabla \otimes \mathbf{u}(\mathbf{x})) \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = 0, \quad \mathbf{x} \in \Sigma^t \\
&\mathbf{u}(\mathbf{x}) \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = 0, \quad \mathbf{n} \cdot (\mathbf{C}{:}\nabla \otimes \mathbf{u}(\mathbf{x})) \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \Sigma^n
\end{aligned} \tag{1.7}$$

where ω is the frequency, γ is the phase, and $\mathbf{u}(\mathbf{x})$ is the characteristic mode of vibration; it is convenient also to introduce the associated characteristic value

$$\lambda = \rho\omega^2, \quad \omega = \sqrt{\lambda/\rho} \tag{1.8}$$

Integrating the second equation of (1.7) over the region B and taking the boundary conditions into account, we obtain

$$\lambda = \langle \nabla \otimes \mathbf{u}{:}\mathbf{C}{:}\nabla \otimes \mathbf{u} \rangle / \langle \mathbf{u} \cdot \mathbf{u} \rangle \tag{1.9}$$

(here and henceforth angle brackets denote integration over the region B).

If the elasticity tensor $\mathbf{C}$ is such that the numerator in (1.9) is positive definite, then all of the characteristic values are positive. Moreover, we know that they form a denumerable set $\lambda_1, \lambda_2, \ldots$, and also

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ \ldots \ \leqslant \lambda_n \leqslant \ \ldots, \quad \lim_{n\to\infty} \lambda_n = +\infty \tag{1.10}$$

In addition to (1.9) and (1.10), the characteristic values and associated modes possess a number of well-known properties, of which we will henceforth need only the so-called minimax property. In order to specify this, we introduce some ideas and notation.

Let $\mathcal{K}$ be the set of all continuous piecewise-smooth vector fields obeying the kinematic boundary conditions, let $\mathcal{W}_n$ be the system of n piecewise-continuous vector fields $\mathbf{w}_i(\mathbf{x})$ ($i = 1, \ldots, n$), and let $\mathcal{W}_n^\perp$ be the orthogonal complement, in the sense of the scalar product $\langle \mathbf{u} \cdot \mathbf{v} \rangle$, of the linear hull of the system $\mathcal{W}_n$ in the space of piecewise-continuous vector fields on B. Then

$$\lambda_{n+1} = \sup_{\{\mathcal{W}_n\}} \ \inf_{\mathcal{W}_n^\perp \cap \mathcal{K}} \frac{\langle \nabla \otimes \mathbf{u}{:}\mathbf{C}{:}\nabla \otimes \mathbf{u} \rangle}{\langle \mathbf{u} \cdot \mathbf{u} \rangle} \tag{1.11}$$

where $\{\mathcal{W}_n\}$ is the set of all $\mathcal{W}_n$.

From (1.11) it follows that, provided the tensors $\mathbf{C}'$ and $\mathbf{C}$ are such that for the same region with the same boundary conditions, the inequality

$$\langle \nabla \otimes \mathbf{u}{:}\mathbf{C}'{:}\nabla \otimes \mathbf{u} \rangle \leqslant \langle \nabla \otimes \mathbf{u}{:}\mathbf{C}{:}\nabla \otimes \mathbf{u} \rangle \tag{1.12}$$

holds for all admissible fields ($\mathbf{u}(\mathbf{x}) \in \mathcal{K}$), then for the characteristic values with the same numbers the following inequality holds

$$\lambda'_n \leqslant \lambda_n, \quad n = 1, 2, \ldots \tag{1.13}$$

If the regions and/or the boundary conditions are different, but such that $\mathcal{K}' \supset \mathcal{K}$ or the fields which belong to $\mathcal{K}$, being continued by zero to the region $B' \supset B$, appear to belong to $\mathcal{K}'$, and additionally for all $\mathbf{u}(\mathbf{x}) \in \mathcal{K}$ inequality (1.12) holds, then for the characteristic values with the same numbers inequality (1.13) also holds.

The assertions stated above, expressed finally in terms of inequalities (1.13), form the basis for a proof of the classical comparison theorem for the characteristic values [3]. Modifications of the comparison theorem for homogeneous bodies under the appropriate boundary conditions (i.e. under the circumstances when Van Hove's theorem or some of its modifications is valid) will be deduced from the same assertions.

2. VAN HOVE'S THEOREM AND ITS MODIFICATIONS

We give, first, some definitions and then state Van Hove's theorem and three of its modifications. The original Van Hove's proof [4] is reproduced in [3]. The proofs of modifications can be found in [5–7].

Definition. The quantities

$$\underline{c} = \min_{|\mathbf{g}_1|=|\mathbf{g}_2|=1} \mathbf{g}_1 \otimes \mathbf{g}_2 : \mathbf{C} : \mathbf{g}_1 \otimes \mathbf{g}_2 \tag{2.1}$$

$$\bar{c} = \max_{|\mathbf{g}_1|=|\mathbf{g}_2|=1} \mathbf{g}_1 \otimes \mathbf{g}_2 : \mathbf{C} : \mathbf{g}_1 \otimes \mathbf{g}_2 \tag{2.2}$$

are called, respectively, the lower and the upper Hadamard numbers of the fourth-rank tensor **C**.

Clearly, the validity of the Hadamard inequality (see [3, 8]) for a fourth-rank tensor is equivalent to non-negativeness of the lower Hadamard number, and the validity of the strong-ellipticity inequality is equivalent to its positiveness. In terms of eigenvalues of the acoustic tensor $\mathbf{A}(\mathbf{n}) = \mathbf{n} \otimes \mathbf{n} : \mathbf{C}^{(1324)}$ [3] for all values of the wave normal **n**, the lower Hadamard number is the minimum of the least eigenvalue with respect to **n**, and the upper Hadamard number is the maximum of the greatest eigenvalue. In other words,

$$\underline{c} = \rho \underline{a}^2, \quad \bar{c} = \rho \bar{a}^2 \tag{2.3}$$

where $\underline{a}$ and $\bar{a}$ are, respectively, the least and the greatest propagation velocities for elastic waves of all possible polarizations, corresponding to various directions of propagation in a solid of the density ρ with the elasticity tensor **C**.

Van Hove's theorem. Let B be a bounded regular region in R^3, let **C** be a constant fourth-rank tensor, and let $\mathbf{u}(\mathbf{x})$ be a continuous piecewise-smooth vector field, which vanishes ∂B. Then the following inequalities hold

$$\underline{c}\langle \nabla \otimes \mathbf{u} : \nabla \otimes \mathbf{u} \rangle \leq \langle \nabla \otimes \mathbf{u} : \mathbf{C} : \nabla \otimes \mathbf{u} \rangle \leq \bar{c} \langle \nabla \otimes \mathbf{u} : \nabla \otimes \mathbf{u} \rangle \tag{2.4}$$

This formulation, which is somewhat different from the original [4], is obviously equivalent to it.

Before stating the modifications of Van Hove's theorem, we define the fourth-rank tensors invariant with respect to one or several specular reflections. For simplicity we will denote the representations of specular reflections in the space of fourth-rank tensors by the same symbols as the reflections themselves (which act in the vector space). We will denote the image of the tensor **C** under the reflection **Q** by $\mathbf{C} * \mathbf{Q}$, where for decomposable tensors (tetrads) this denotes the following

$$\mathbf{c}_1 \otimes \mathbf{c}_2 \otimes \mathbf{c}_3 \otimes \mathbf{c}_4 * \mathbf{Q} = \mathbf{c}_1 \cdot \mathbf{Q} \otimes \mathbf{c}_2 \cdot \mathbf{Q} \otimes \mathbf{c}_3 \cdot \mathbf{Q} \otimes \mathbf{c}_4 \cdot \mathbf{Q} \tag{2.5}$$

which for an arbitrary tensor each tetrad is converted by (2.5) into its representation.

The second-rank tensor

$$\mathbf{Q}_i = \mathbf{I} - 2\mathbf{e}_i \otimes \mathbf{e}_i \tag{2.6}$$

specifies reflection with respect to the plane with normal $\mathbf{e}_i$.

We call the fourth-rank tensor **C** specularly symmetric with respect to the plane with normal $\mathbf{e}_i$, if

$$\mathbf{C}*\mathbf{Q}_i = \mathbf{C} \tag{2.7}$$

We call the fourth-rank tensor **C** orthotropic, if a triplet of mutually orthogonal planes exists, with respect to which it is specularly symmetric

$$\mathbf{C}*\mathbf{Q}_i = \mathbf{C}, \quad i = 1, 2, 3, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{2.8}$$

These planes are referred to as planes of orthotropy in this case.

Modified Van Hove's theorem 1. Let B be a rectangular parallelepiped with faces having unit normals $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$, $\pm\mathbf{e}_3$ (the first, second and third pairs of faces), let **C** be a constant fourth-rank tensor and let $\mathbf{u}(\mathbf{x})$ be a continuous piecewise-smooth vector field.

Then

1. if the tensor **C** is specularly symmetric with respect to the plane of some pair of faces (say, the first one) and if $\mathbf{u}(\mathbf{x})$ satisfies the tangentiality condition ($\mathbf{u} \cdot \mathbf{e}_1 = 0$) on the faces of this pair and vanishes on the others, then inequality (2.4) holds;

2. if the tensor **C** is orthotropic with orthotropy planes parallel to the planes of the faces and if $\mathbf{u}(\mathbf{x})$ satisfies the tangentiality condition on all faces ($\mathbf{u} \cdot \mathbf{e}_i = 0$ on the faces of the ith pair), then inequality (2.4) also holds.

Modified Van Hove's theorem 2. This repeats both assertions of the modified Van Hove's theorem 1 apart from the tangentiality condition for the field $\mathbf{u}(\mathbf{x})$ that is replaced by the condition of its normality on the faces of one pair or on all the faces, respectively.

Modified Van Hove's theorem 3. This embraces various mixed cases, when conditions of different types (tangentiality or normality) are specified on different pairs of faces or on different faces of the same pair.

3. MODIFICATIONS OF THE COMPARISON THEOREM AND BILATERAL BOUNDS FOR THE CHARACTERISTIC VALUES

Note, that if the region B, the boundary conditions and strongly elliptic fourth-rank tensors $\mathbf{C}'$ and **C** obey the hypothesis of some theorem of Van Hove's type, and for biquadratic forms of the tensors $\mathbf{C}'$ and **C** the inequality

$$\mathbf{f}\otimes\mathbf{g}: \mathbf{C}': \mathbf{f}\otimes\mathbf{g} \leqslant \mathbf{f}\otimes\mathbf{g}: \mathbf{C}: \mathbf{f}\otimes\mathbf{g}, \quad \forall \mathbf{f}, \mathbf{g} \tag{3.1}$$

holds, then for non-zero admissible vector fields $\mathbf{u}(\mathbf{x})$ we have

$$0 < \langle \nabla\otimes\mathbf{u}: \mathbf{C}': \nabla\otimes\mathbf{u} \rangle \leqslant \langle \nabla\otimes\mathbf{u}: \mathbf{C}: \nabla\otimes\mathbf{u} \rangle \tag{3.2}$$

which is identical with inequality (1.12) (it follows from inequality (2.4) for the difference of the tensors **C** and $\mathbf{C}'$).

Let us return to the free vibration problem for homogeneous elastic bodies. From the positive definiteness of the quadratic functionals in (3.2) it follows that free vibrations do exist, and from (1.12) we obtain inequalities (1.13) for positive characteristic values, when two bodies with the elasticity tensors **C** and $\mathbf{C}'$, respectively, occupy the same region having the shape of a rectangular parallelepiped, with the same boundary conditions corresponding to some theorem of Van Hove's type.

Suppose that the regions occupied by the bodies have a different shape, and moreover are possibly different, or portions of the boundary where conditions of different types are specified (zero displacements, tangentiality or normality with respect to the boundary surface) are different. Suppose additionally that each of the fields can be continued by zero up to the field on a rectangular parallelepiped, that obeys the hypothesis of some theorem of Van Hove's type, and moreover, all the fields, admissible for the body with elasticity tensor **C**, are simultaneously admissible for the body with the elasticity tensor $\mathbf{C}'$. Then the validity of inequalities (3.1), implies the validity of inequalities (1.13) for characteristic values (see Section 1).

The assertions stated above represent the totality of modifications of the classical comparison theorem for free vibrations. In particular, if the boundary is clamped, then apart from the validity of inequality (3.1), it is only necessary that the region B be a subregion of B' (it is sufficient to employ the original Van Hove's theorem here).

The difference between the established modifications of the comparison theorem and the classical one is that the classical theorem requires an inequality of the type of (3.1) to hold not only for the dyads $\mathbf{f} \otimes \mathbf{g}$, but also for arbitrary second-rank tensors, which is a much stronger condition than (3.1).

We will illustrate this difference by the example of bodies which obey Hooke's law with shear and bulk moduli G', K' and G, K, respectively. The classical comparison theorem requires that the inequalities

$$0 < G' \leqslant G, \quad 0 < K' \leqslant K$$

hold simultaneously, whereas inequality (3.1) is equivalent to the system of inequalities

$$0 < G' \leqslant G, \quad 0 < K' + \frac{4}{3}G' \leqslant K + \frac{4}{3}G$$

Thus, K' can be significantly greater than K provided that G' is significantly smaller than G; for a sufficiently large difference between the shear moduli, the quantity K may even be negative, and still for such a body (provided that it obeys the hypothesis of some theorem of Van Hove's type) the characteristic values, and hence, frequencies with the same numbers, will satisfy inequality (1.13).

We will consider the problem of obtaining bilateral bounds for characteristic values, and hence, for corresponding frequencies. We will concentrate first, on the case when the boundary of a homogeneous body that occupies the region B, is clamped. Along with a given body having strongly elliptic elasticity tensor $\mathbf{C}$ we introduce two fictitious "comparison bodies" with clamped boundary, occupying the regions $B^{(1)}$ and $B^{(2)}$ ($B^{(1)} \supset B \supset B^{(2)}$) and having the "elasticity tensors" $\mathbf{C}^{(1)} = \underline{c}\mathbf{1}$ and $\mathbf{C}^{(2)} = \underline{c}\mathbf{1}$, respectively, where $\underline{c} < 0$ and $\bar{c} < 0$ are the lower and the upper Hadamard numbers of the tensor $\mathbf{C}$ and, $\mathbf{1} = \mathbf{1} \otimes \mathbf{I}^{(1324)}$ is the unit fourth rank tensor (the identity operator in the space of second-rank tensors). Clearly, the inequalities

$$\mathbf{f} \otimes \mathbf{g}{:}\mathbf{C}^{(1)}{:}\, \mathbf{f} \otimes \mathbf{g} \leqslant \mathbf{f} \otimes \mathbf{g}{:}\, \mathbf{C}{:}\, \mathbf{f} \otimes \mathbf{g} \leqslant \mathbf{f} \otimes \mathbf{g}{:}\, \mathbf{C}^{(2)}{:}\, \mathbf{f} \otimes \mathbf{g}, \quad \forall \mathbf{f}, \mathbf{g}$$

then hold. Hence, for the corresponding characteristic members the following inequalities hold

$$0 < \lambda_n^{(1)} \leqslant \lambda_n \leqslant \lambda_n^{(2)}, \quad n = 1, 2, \ldots \tag{3.3}$$

i.e. the numbers $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ yield a bilateral bound for λ_n. It remains only to find their explicit values. Note, that the free vibration equations (1.7) for the comparison bodies, are the characteristic equations for the Laplace operator with Dirichlet boundary conditions. It is obvious that each of them is equivalent to a system of three scalar Helmholtz equations with the same boundary conditions. In other words, every characteristic function associated with the Laplace operator (a solution of the Helmholtz equation) generates a three-dimensional subspace of characteristic vector-valued functions with the same characteristic value; $\lambda^{(1)}$ is obtained from the characteristic value by multiplying by $\underline{c}$, and $\lambda^{(2)}$ by multiplying by $\bar{c}$. Thus, the numbers $\lambda^{(1)}$ and $\lambda^{(2)}$ are triply degenerate (provided that the characteristic values of the scalar Laplace operator are non-degenerate).

If the region B is sufficiently simple and the characteristic values M_i, $i = 1, 2, \ldots$, of the Laplace operator are known for it, then it is possible to set $B^{(1)} = B^{(2)} = B$ and we obtain

$$\lambda_{3i-2}^{(1)} = \lambda_{3i-1}^{(1)} = \lambda_{3i}^{(1)} = \underline{c}\mu_i, \quad \lambda_{3i-2}^{(2)} = \lambda_{3i-1}^{(2)} = \lambda_{3i}^{(3)} = \bar{c}\mu_i, \quad i = 1, 2, \ldots$$

If the values μ_i for the region B are unknown, some standard regions, say, parallelepipeds with dimensions $l_1^{(k)}, l_2^{(k)}, l_3^{(k)}$ ($k = 1, 2$), may be used as $B^{(1)}$ and $B^{(2)}$, in addition it is necessary to satisfy the relation $B^{(1)} \supset B \supset B^{(2)}$. It then becomes possible to make use of the well-known solution for a parallelepiped with the Dirichlet condition

$$\mu_i^{(k)} = \pi^2 \left(\frac{(n_1^{(k)})^2}{(l_1^{(k)})^2} + \frac{(n_2^{(k)})^2}{(l_2^{(k)})^2} + \frac{(n_3^{(k)})^2}{(l_3^{(k)})^2} \right), \quad n_1^{(k)},\ n_2^{(k)},\ n_3^{(k)} = 1, 2, \ldots, \quad k = 1, 2$$

where $\mu_i^{(k)}$ for each value of k are assumed to be numbered in i in the form of a non-decreasing sequence.

To obtain somewhat physically clear bounds for the first frequency (the lowest in the spectrum), we may take $\mu_1^{(1)}$ for a parallelepiped $B^{(1)}$ that contains B, and $\mu_1^{(2)}$, say, for a sphere $B^{(2)}$ that lies inside B

$$\mu_1^{(1)} = \pi^2\left(\frac{1}{(l_1^{(1)})^2}+\frac{1}{(l_2^{(1)})^2}+\frac{1}{(l_3^{(1)})^2}\right) > \frac{\pi^2}{(l_1^{(1)})^2}, \quad \mu_1^{(2)} = \frac{4\pi^2}{(l^{(2)})^2}$$

where $l^{(2)}$ is the diameter of sphere $B^{(2)}$, and then optimize them in a reasonable way. The best upper bound of such a type corresponds to an inscribed sphere of the greatest possible diameter, which we denote by $l_*^{(2)}$. As for a parallelepiped, we do the following: let $l_{1*}^{(1)}$ be the least thickness of a layer that contains B, $l_{2*}^{(1)}$ the least thickness of a layer that contains B and whose plane is orthogonal to the plane of the first layer, $l_{3*}^{(1)}$ the least thickness of a layer that contains B and whose plane is orthogonal to the planes of two preceding ones. Then

$$\pi\frac{\underline{a}}{l_{1*}^{(1)}} < \pi\frac{\underline{a}}{l_{1*}^{(1)}}\left(1+\left(\frac{l_{1*}^{(1)}}{l_{2*}^{(1)}}\right)^2+\left(\frac{l_{1*}^{(1)}}{l_{3*}^{(1)}}\right)^2\right)^{1/2} \leqslant \omega_1 \leqslant 2\pi\frac{\bar{a}}{l_*^{(2)}} \tag{3.4}$$

These bounds are simply related to the times of propagation of both the fastest and the slowest waves for distances which are characteristic geometrical parameters of the region B.

Note that bilateral bounds are found for a homogeneous body of arbitrary anisotropy, when an analytical solution is unknown even for standard regions (spheres or parallelepipeds); however, the lower and the upper Hadamard numbers can be found either analytically or at least numerically, and this problem of finding an extremum for a function is, in principle, quite simple.

If we consider the case when the region B is a parallelepiped with, say, conditions of, sliding all over the boundary, and the elasticity tensor $\mathbf{C}$ is orthotropic with the orthotropy planes parallel to the planes of the faces. Then, taking account of the isotropy of the tensors $\underline{c}\mathbf{1}$, $\bar{c}\mathbf{1}$ and making use of the modified Van Hove's theorem 1, we arrive at inequalities (3.3), where $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ are the characteristic values of the following boundary-value problems

$$c^{(k)}\Delta\mathbf{u}^{(k)}(\mathbf{x})+\lambda^{(k)}\mathbf{u}^{(k)}(\mathbf{x}) = 0, \quad \begin{cases}\mathbf{u}^{(k)}\cdot\mathbf{n}|_{\partial B} = 0\\ \mathbf{n}\cdot\nabla\otimes\mathbf{u}^{(k)}\cdot(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})|_{\partial B} = 0, \quad k = 1,2,\end{cases}$$

$$c^{(1)} = \underline{c}, \quad c^{(2)} = \bar{c}$$

and are numbered in non-decreasing order. The solutions of such problems are well-known: introducing axes parallel to the edges of the parallelepiped, we split each of the above two problems into three problems for the components. For instance, for $u_1^{(1)}(\mathbf{x})$ we have

$$\underline{c}\Delta u_1^{(1)} + \lambda^{(1)}u_1^{(1)} = 0$$

$$u_1^{(1)}|_{x_1=0} = u_1^{(1)}|_{x_1=l_1} = 0; \quad \left.\frac{\partial u_1^{(1)}}{\partial x_2}\right|_{x_2=0} = \left.\frac{\partial u_1^{(1)}}{\partial x_2}\right|_{x_2=l_2} = \left.\frac{\partial u_1^{(1)}}{\partial x_3}\right|_{x_3=0} = \left.\frac{\partial u_1^{(1)}}{\partial x_3}\right|_{x_3=l_3} = 0$$

Hence

$$\mu_i = \pi^2\left(\frac{n_1^2}{l_1^2}+\frac{n_2^2}{l_2^2}+\frac{n_3^2}{l_3^2}\right), \quad n_1, n_2, n_3 = 0,1,2,\ldots; \quad n_1^2+n_2^2+n_3^2 \geqslant 1$$

where μ_i are numbered in non-decreasing order, taking into account the degeneration multiplicity; it is the values of μ_i corresponding to the cases when only one of the numbers n_k is non-zero (or two of the numbers or all three numbers), that are considered as single (double and triple, respectively). Here, the relation $\lambda_i^{(k)} = c^{(k)}\mu_i$ holds.

If $l_1 \leqslant l_2 \leqslant l_3$, then

$$\mu_1 = \frac{\pi^2}{l_3^2}; \quad \underline{c}\frac{\pi^2}{l_3^2} \leqslant \lambda_1 \leqslant \bar{c}\frac{\pi^2}{l_3^2} \Leftrightarrow \pi\frac{\underline{a}}{l_3} \leqslant \omega_1 \leqslant \pi\frac{\bar{a}}{l_3}$$

where the bounds for ω_1 are again simply related to the times of propagation of both the fastest and the slowest waves over a distance equal to the length of the longest edge of the parallelepiped.

Consider a number example. Let $l_3 = 5l_2 \geqslant 5l_1$, $\bar{c} = 3/2\underline{c}$. Then

$$\mu_k = \pi^2 k^2 / l_3^2, \quad k = 1, \ldots, 5$$

and moreover, there is no degeneration for $k = 1, \ldots, 4$; it begins only with $k = 5$ ($\mu_5 = \mu_6$). For the intervals containing the first four values of λ_k, measured in units of $\pi^2\underline{c}/l_3^2$, we have

$$1 \leqslant \lambda_1 \leqslant 1.5, \quad 4 \leqslant \lambda_2 \leqslant 6, \quad 9 \leqslant \lambda_3 \leqslant 13.5, \quad 16 \leqslant \lambda_4 \leqslant 24, \quad 25 \leqslant \lambda_5 \leqslant 37.5$$

i.e. the intervals do not overlap with each other and with the interval for λ_5 and λ_6; they yield completely effective bilateral bounds. Note, that the bounds hold for an arbitrary orthotropic body (appropriately oriented), i.e. in the case when in general it seems impossible to find an analytical solution.

4. SOME MODIFICATIONS OF THE PROPOSED METHOD IN THE CASE OF UNIFORMLY STRESSED NON-LINEARLY ELASTIC BODIES AND MODERATELY INHOMOGENEOUS BODIES

The linear equation of motion (1.5) can be obtained by linearizing the non-linear equation with respect to the displacement gradient in the case when the latter is small. We will assume that an initially strained non-linearly elastic body is in equilibrium under certain boundary conditions. We will consider the vibrations corresponding to the small-gradient displacements with respect to the equilibrium configuration $\varkappa$ (taken as a reference one). In the equilibrium configuration, the initial Cauchy stress field $\mathbf{T}_0(\mathbf{x})$ satisfies the equilibrium equation both inside the body and on its boundary over portions of tangential or normal sliding. Note that in the reference configuration the Cauchy stress $\mathbf{T}$ and the Piola stress $\mathbf{T}_\varkappa$ coincide with each other, but in configurations different from the reference one there is no longer such coincidence [8]. We specify the linearized incremental defining relation for the Piola stress tensor by the equality

$$\mathbf{T}_\varkappa(\mathbf{x}, t) - \mathbf{T}_0(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \nabla \otimes \mathbf{u}(\mathbf{x}, t)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the above-mentioned small-gradient displacement field with respect to the reference configuration. Then, the equation of motion takes the form (1.5), where by ρ we mean the reference-configuration density. Provided that the homogeneous non-linearly elastic body is uniformly strained in the reference configuration, the tensor $\mathbf{C}(\mathbf{x}) = \text{const}$, and all the results of Section 3 remain valid.

Note that the elasticity tensor $\mathbf{C}$ possesses property (1.4); in addition, it is related specifically to some different elasticity tensor $\mathbf{L}$ that specifies the linearized relation between the Jaumann increment of the Cauchy stress tensor and the small-strain tensor (the symmetrized gradient of the displacement field) [8]. Since the tensor $\mathbf{L}$ is a conventional characteristic of incremental elasticity, let us compare the lower and the upper Hadamard numbers $\underline{c}$ and $\bar{c}$ with the corresponding quantities l_- and l^-.

First, from the relation between $\bar{\mathbf{C}}$ and $\mathbf{L}$

$$\mathbf{C} = \mathbf{L} + \mathbf{T}_0 \otimes \mathbf{I} + \frac{1}{2}(\mathbf{T}_0 \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{T}_0)^{(1423)} - \frac{1}{2}(\mathbf{T}_0 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T}_0)^{(1432)}$$

we find, that for the unit vectors $\mathbf{f}$ and $\mathbf{g}$

$$\mathbf{f} \otimes \mathbf{g} : \mathbf{C} : \mathbf{f} \otimes \mathbf{g} = \mathbf{f} \otimes \mathbf{g} : \mathbf{L} : \mathbf{f} \otimes \mathbf{g} + \frac{1}{2}\mathbf{T}_0' : (\mathbf{f} \otimes \mathbf{f} - \mathbf{g} \otimes \mathbf{g})$$

where $\mathbf{T}_0' = \mathbf{T}_0 - 1/3\mathbf{I}(\mathbf{I}: \mathbf{T}_0)$ is the Cauchy stress deviator. Then, using the Cauchy–Bunyakovskii inequality for the tensors $\mathbf{T}_0'$ and $(\mathbf{f} \otimes \mathbf{f} - \mathbf{g} \otimes \mathbf{g})$, we get

$$\mathbf{f} \otimes \mathbf{g}:\mathbf{L}:\mathbf{f} \otimes \mathbf{g} - \tau_0 \leqslant \mathbf{f} \otimes \mathbf{g}:\mathbf{C}:\mathbf{f} \otimes \mathbf{g} \leqslant \mathbf{f} \otimes \mathbf{g}:\mathbf{L}:\mathbf{f} \otimes \mathbf{g} + \tau_0$$

where $\tau_0 = (\mathbf{T}_0':\mathbf{T}_0'/2)^{1/2}$ is the shear stress intensity. Hence we obtain the inequalities

$$\underline{l} - \tau_0 \leqslant \underline{c} \leqslant \underline{l} + \tau_0, \quad \bar{l} - \tau_0 \leqslant \bar{c} \leqslant \bar{l} + \tau_0 \tag{4.1}$$

which give effective estimates $\underline{c}$ and $\bar{c}$ when the shear stresses are not large compared with the elastic moduli. If we substitute $\underline{l} - \tau_0$ for $\underline{c}$ and $\bar{l} + \tau_0$ for $\bar{c}$ in the bilateral bounds for the characteristic values, we again arrive at rigorous bilateral bounds, albeit slightly rougher.

We will now consider the case when the elasticity tensor field $\mathbf{C}(\mathbf{x})$ is not uniform, but (in the sense clarified below) does not differ greatly from some uniform field $\mathbf{C}_0$. We will show how to reduce this case to the uniform case.

Introducing a certain quantity

$$\alpha_0 = \sup_{\mathbf{x} \in B}((\mathbf{C}(\mathbf{x}) - \mathbf{C}_0) \vdots (\mathbf{C}(\mathbf{x}) - \mathbf{C}_0))^{1/2}$$

which characterize the deviation of $\mathbf{C}(\mathbf{x})$ from $\mathbf{C}_0$, for an arbitrary second-rank tensor $\mathbf{H}$ we obtain

$$\mathbf{H}:(\mathbf{C}_0 - \alpha_0 \mathbf{1}):\mathbf{H} \leqslant \mathbf{H}:\mathbf{C}:\mathbf{H} \leqslant \mathbf{H}:(\mathbf{C}_0 + \alpha_0 \mathbf{1}):\mathbf{H}$$

(by virtue of the Cauchy–Bunyakovskii inequality for the tensors $\mathbf{C} - \mathbf{C}_0$ and $\mathbf{H} \otimes \mathbf{H}$).

Further using the classical comparison theorem, we find that the characteristic values for the body with the elasticity tensor $\mathbf{C}$ are contained in the interval between characteristic values with the same numbers for fictitious homogeneous bodies with elasticity tensors $\mathbf{C}_0^- = \mathbf{C}_0 - \alpha_0\mathbf{1}$ and $\mathbf{C}_0^+ = \mathbf{C}_0 + \alpha_0\mathbf{1}$, respectively. Note, that

$$\underline{c}_0^- = \underline{c}_0 - \alpha_0; \quad \bar{c}_0^+ = \bar{c}_0 + \alpha_0$$

and for each of the comparison bodies we may use the results of Section 3, provided that

$$\underline{c}_0^- > 0 \Leftrightarrow \alpha_0 < \underline{c}_0 \tag{4.2}$$

Inequality (4.2) gives a rigorous meaning to the supposition of moderate non-uniformity of the field $\mathbf{C}(\mathbf{x})$. Unlike any version of perturbation theory, it is not necessary here that the value of α_0 should be extremely small.

5. AN EXAMPLE OF A COMPARISON OF BOUNDS OF THE SAME TYPE BASED ON THE CLASSICAL COMPARISON THEOREM AND ON ITS MODIFICATIONS

Let the elasticity tensor $\mathbf{C}$ be constant, and let the region B occupied by the body be a rectangular parallelepiped $0 \leqslant \mathbf{x} \cdot \mathbf{e}_i \leqslant l_i$ $(i = 1, 2, 3)$, $l_1 < l_2 \leqslant l_3$, on the boundary of which the body is clamped. Then, by virtue of the first modification of the classical theorem (Section 3) for the first characteristic value the following inequality holds

$$\frac{\pi^2 \underline{c}}{l_1^2} < \pi^2 \underline{c}\left(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2}\right) \leqslant \lambda_1 \tag{5.1}$$

On the other hand, by (1.11) for $n = 0$

$$\lambda_1 = \inf_{\mathscr{K}} \frac{\langle \nabla \otimes \mathbf{u}:\mathbf{C}:\nabla \otimes \mathbf{u} \rangle}{\langle \mathbf{u} \cdot \mathbf{u} \rangle} \tag{5.2}$$

It follows from (5.2) that the ratio on the right-hand side of the equality, for any kinematically admissible field, $\mathbf{u}(\mathbf{x})$ is an upper bound for the characteristic value λ_1. In particular, we can take as such a field

$$\mathbf{u}(\mathbf{x})=\mathbf{g}\sin\frac{\pi x_1}{l_1}\sin\frac{\pi x_2}{l_2}\sin\frac{\pi x_3}{l_3},\quad x_i=\mathbf{x}\cdot\mathbf{e}_i,\ |\mathbf{g}|=1 \tag{5.3}$$

Then

$$\lambda_1\leq\pi^2\left(\frac{\mathbf{e}_1\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_1\otimes\mathbf{g}}{l_1^2}+\frac{\mathbf{e}_2\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_2\otimes\mathbf{g}}{l_2^2}+\frac{\mathbf{e}_3\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_3\otimes\mathbf{g}}{l_3^2}\right) \tag{5.4}$$

Let the orientation of the parallelepiped with respect to characteristic directions in an anisotropic solid, be such that the vector $\mathbf{e}_1$ is collinear with the normal to the front of the slowest elastic wave. Taking into account that $\mathbf{g}$ in (5.3) and (5.4) is an arbitrary unit vector, we choose it to coincide with the polarization vector of that wave. Then $\mathbf{e}_1\otimes\mathbf{g}$ is the dyad on which the biquadratic form (2.1) attains its minimum value

$$\underline{c}=\mathbf{e}_1\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_1\otimes\mathbf{g} \tag{5.5}$$

By virtue of (5.5) inequalities (5.1) and (5.4) take the form

$$\pi^2\underline{c}\left(\frac{1}{l_1^2}+\frac{1}{l_2^2}+\frac{1}{l_3^2}\right)\leq\lambda_1\leq\pi^2\underline{c}\left(\frac{1}{l_1^2}+\frac{\mathbf{e}_2\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_2\otimes\mathbf{g}}{\underline{c}}\frac{1}{l_2^2}+\frac{\mathbf{e}_3\otimes\mathbf{g}:\mathbf{C}:\mathbf{e}_3\otimes\mathbf{g}}{\underline{c}}\frac{1}{l_3^2}\right) \tag{5.6}$$

If the parallelepiped is considerably flattened in the direction $\mathbf{e}_1$ ($l_1\ll l_2$ and $l_1\ll l_3$), then the upper and lower bounds in (5.6) almost merge and differ as little as desired from the quantity $\pi^2\underline{c}/l_1^2$. In other words, for the chosen orientation of the parallelepiped, the lower bound for the lowest characteristic frequency, based on a modified comparison theorem, approaches its exact value in the limit of small thicknesses.

Note, that the bound (5.1), obtained by means of the comparison body with elasticity tensor $\underline{c}\mathbf{1}$, can be considered as found using a certain Hookean comparison body with the elasticity tensor $\underline{c}(2\mathbf{1}^{\text{def}}-\mathbf{I}\otimes\mathbf{I})$, where $\mathbf{1}^{\text{def}}$ is the orthogonal projector on the subspace of symmetric second-rank tensors

$$\mathbf{1}^{\text{def}}=\frac{1}{2}(\mathbf{I}\otimes\mathbf{I}^{(1324)}+\mathbf{I}\otimes\mathbf{I}^{(1342)})$$

$$\mathbf{1}^{\text{def}}:\mathbf{H}=\mathbf{H}^s=\frac{1}{2}(\mathbf{H}+\mathbf{H}^T),\quad \mathbf{H}:\mathbf{1}^{\text{def}}:\mathbf{H}=\mathbf{H}^s:\mathbf{H}^s$$

Indeed, for any dyad $\mathbf{H}$ we have

$$\mathbf{f}\otimes\mathbf{g}:\mathbf{1}:\mathbf{f}\otimes\mathbf{g}=\mathbf{f}\otimes\mathbf{g}:(2\mathbf{1}^{\text{def}}-\mathbf{I}\otimes\mathbf{I}):\mathbf{f}\otimes\mathbf{g}$$

and, hence, by any of the Van Hove theorems

$$\langle\nabla\otimes\mathbf{u}:\nabla\otimes\mathbf{u}\rangle=\langle\nabla\otimes\mathbf{u}:\mathbf{1}:\nabla\otimes\mathbf{u}\rangle=\langle\nabla\otimes\mathbf{u}:(2\mathbf{1}^{\text{def}}-\mathbf{I}\otimes\mathbf{I}):\nabla\otimes\mathbf{u}\rangle$$

This means that under appropriate conditions (say, clamping over the whole boundary) the characteristic frequencies and modes both for the body with elasticity tensor $\underline{c}\mathbf{1}$ and for the body with elasticity tensor $\underline{c}(2\mathbf{1}^{\text{def}}-\mathbf{I}\otimes\mathbf{I})$ are the same; despite the fact that the second body possesses a negative bulk modulus, contrary to conventional opinion, it is stable under conditions of clamping, and it can execute free vibrations.

We will now show that in the same problem with some special orthotropic tensor $\mathbf{C}$ the best lower bound for the lowest frequency, found on the basis of the classical theorem by means of the Hookean comparison bodies (i.e. a bound of the same type), can be as small as desired with respect to the exact value.

Suppose the elasticity tensors of the orthotropic body $\mathbf{C}$ and the Hookean comparison body $\mathbf{C}'$ are specified by the equalities [9]

$$\mathbf{C}=2G((\mathbf{1}^{\text{def}}-\mathbf{S}\otimes\mathbf{S})+h\mathbf{S}\otimes\mathbf{S}),\quad \mathbf{S}:\mathbf{I}=0,\quad \mathbf{S}=\mathbf{S}^T,\quad \mathbf{S}:\mathbf{S}=1,\quad h<1 \tag{5.7}$$

$$\mathbf{C}'=2G'\left(\mathbf{1}^{\text{def}}-\frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)+K'\mathbf{I}\otimes\mathbf{I}$$

By the condition of the classical comparison theorem

$$\mathbf{H}:\mathbf{C}':\mathbf{H}\leq\mathbf{H}:\mathbf{C}:\mathbf{H},\quad\forall\mathbf{H}$$

whence we obtain the following inequality for the moduli of the Hookean body

$$G' \leqslant Gh, \quad K' \leqslant \frac{2}{3}G \tag{5.8}$$

Instead of finding the exact value of the quantity λ'_1 for the Hookean body (that has the same shape and is under the same boundary conditions of clamping as the orthotropic body), we make use of the upper bound (5.4), having replaced the elasticity tensor $\mathbf{C}$ by $\mathbf{C}'$ and the vector $\mathbf{g}$ and $\mathbf{e}_3$. Then the upper bound for λ'_1 takes the form

$$\lambda_1' \leqslant \pi^2 \left(\frac{G'}{l_1^2} + \frac{G'}{l_2^2} + \frac{K' + \frac{4}{3}G'}{l_3^2} \right) \leqslant \pi^2 \left(\frac{Gh}{l_1^2} + \frac{Gh}{l_2^2} + \frac{2}{3}\frac{G(1+2h)}{l_3^2} \right)$$

which in the limit of small thicknesses reduces to

$$\lambda_1' \leqslant \pi^2 \frac{Gh}{l_1^2} \tag{5.9}$$

We will present without calculations (which are rather complicated) the value of lower Hadamard number for the tensor $\mathbf{C}$ (5.7)

$$\underline{c} = 2G \frac{s_2^2 + h(1 - s_2^2)}{2 - s_2^2 + hs_2^2} \tag{5.10}$$

where s_2 is the second of the eigenvalues of the tensor $\mathbf{S}$, numbered in non-decreasing order.

If $s_2 \neq 0$, i.e. the tensor $\mathbf{S}$ differs from a pure-shear deviator, when $h \ll s_2^2$ equality (5.10) yields

$$\underline{c} \cong G \frac{s_2^2}{1 - s_2^2/2} \gg Gh$$

Hence, in the limit of both small thicknesses and small values of h, the lower bound (5.9) based on the classical theorem, can be as small as desired compared with the exact value, specified by the system of inequalities (5.6), which in the limit considered converts into an equality.

Note, that the above comparison of the bounds holds not only for a parallelepiped, but also for a properly oriented thin disk of arbitrary shape in a plane, for a thin ellipsoid and for a number of other thin clamped bodies.

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